Intro to Circle Geometry
By Raymond Cheong

Many problems involving circles can be solved by constructing right triangles then using the Pythagorean Theorem. The main challenge is identifying where to construct the right triangle. As a hint, sides of the triangle are often:

- A line segment **connecting the center of two circles** that are tangent. This line passes through the tangent point and its length is the sum or the difference of the radii, depending on whether the circles are externally or internally tangent, respectively.

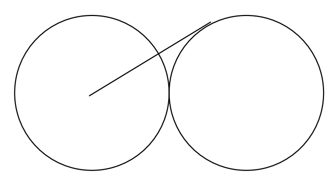
- A **radius to a tangent point**. The radius and the tangent form a right angle.

- **A line parallel or perpendicular to existing lines**, especially to sides of squares or rectangles. This helps to generate right angles.

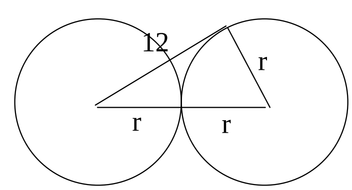
After constructing the triangle, label the sides in terms of, say, the unknown radius r. Then use the **Pythagorean Theorem** to solve for r.

There are usually a few problems every year in the Maryland Math League that can be solved in this way. Consider the following example, which was slightly modified from the 2000-2001 MML (problem 6-4):

<u>Problem</u>: Two externally tangent circles are congruent. A line segment is drawn from the center of one circle, tangent to the other. If the length of this segment is 12, what is the radius of one of the circles?

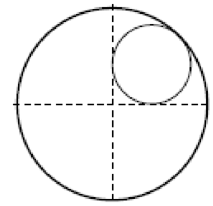


<u>Solution</u>: Create the right triangle shown. One leg is the tangent with length 12. One leg is a radius to the tangent, and denote its length r. The hypotenuse connects the centers of the circle and its length is 2r. By the Pythagorean Theorem, then, $12^2 + r^2 = 2r^2$, giving r = <u>**12**</u>.

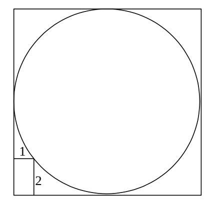


Problems

1) Two perpendicular lines, intersecting at the center of a circle of radius 1, divide the circle into four parts. A smaller circle is inscribed in one of those parts as shown. What is the radius of the smaller circle?

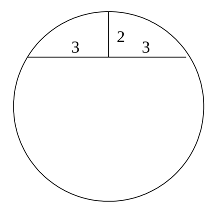


2) A point on a circle inscribed in a square is 1 and 2 units from the two closest sides of the square. Find the area of the square.

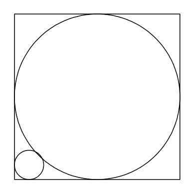


3) A circle of radius 9 is externally tangent to a circle of radius 16. Find the length of their common external tangent.

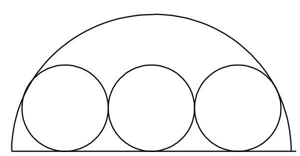
4) In the figure shown, a chord of length 6 is perpendicularly bisected by a line segment of length 2. Find the radius of the circle.



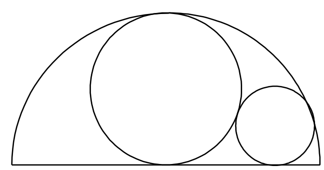
5) In the figure, a circle of radius 1 is inscribed in a square. A smaller circle is tangent to two sides of the square and the first circle. Determine the radius of the smaller circle.



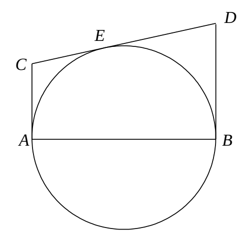
6) Three congruent circles are placed inside a semicircle such they are tangent to the base of the semicircle and to each other as shown. If the semicircle has radius 4, find the radius of one of the circles.



7) As shown in the figure, a circle of radius 2 is tangent to a semicircle and the center of its base. A smaller circle is tangent to the circle, semicircle, and its base. Determine the radius of the smaller circle.

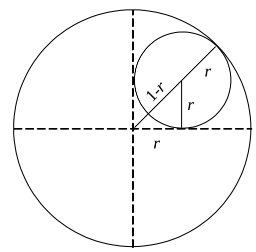


8) In the figure shown, AB is a diameter of a circle, and AC, CD, and BD are tangent to the circle at points A, E, and B, respectively. If CE·DE = 3, determine the area of the circle.



Solutions

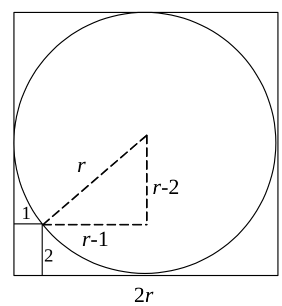
1) [2003 TAMU DE #11]



The key insight in this problem is to use the line connecting the centers of the two circles as the hypotenuse of a right triangle.

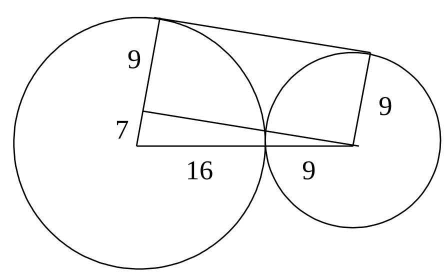
Let r be the radius of the smaller circle. Consider the radius of the large circle that passes through the center of the smaller circle. This radius has length 1, and is split into segments of r and 1-r. Therefore, the right triangle in question has two legs r and hypotenuse 1-r, giving $r^2 + r^2 = (1-r)^2$. Solve to get r = $\boxed{\sqrt{2}-1}$ (the other root is negative).

2) [2001 HMMT General #8]



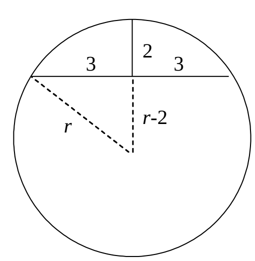
Use the radius to the point in question as the hypotenuse of a right triangle, and use legs parallel to the sides of the squares. If the radius is r, then the legs are r-1 and r-2. Thus, $(r-1)^2 + (r-2)^2 = r^2$, and simplify to $(r-1)(r-5) = 0$. If r = 1 then the square is 2×2 and a point on the circle could not possibly be a distance 2 from an edge. So, r = 5. The area is $(2r)^2 = 10^2 =$ **<u>100</u>**.

3) [Classic]



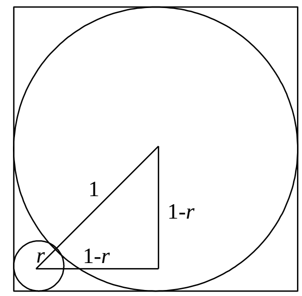
We will use the line segment connecting the centers of the circles as the hypotenuse of a right triangle. To construct this triangle, begin by drawing radii to the points of tangency. The radii meet the external tangent at right angles. Next, draw a line parallel to the external tangent from the center of the smaller circle to the radius of the larger circle. This creates a rectangle and a right triangle. The right triangle has hypotenuse of 16 + 9 = 25 and a leg of 16 – 9 = 7. So the other leg is **<u>24</u>** (a 7-24-25 triangle), which is equal to the length of the external tangent.

4) [Modified from 2002 Tennessee Math Contest Fermat I #20]



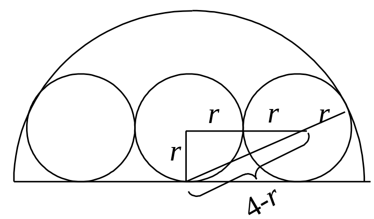
Construct the right triangle as shown, with a radius *r* as the hypotenuse and half of the chord, 3, as a leg. The other leg is the same as a radius minus the perpendicular bisector, or *r*-2. So, $3^2 + (r-2)^2 = r^2$. Solve to get r = **<u>13/4</u>**.

5) [Modified from 2004 UNCW #21]



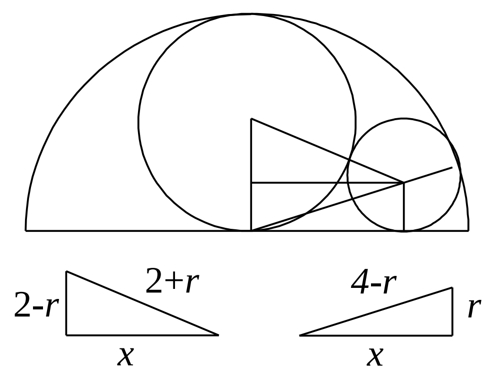
Use the line segment connecting the centers of the circles as the hypotenuse of a right triangle. Draw the legs parallel to the sides of the squares. If we denote the radius of the smaller circle as r, then the hypotenuse has length r+1. Since the center of the smaller circle is a distance r from the two closest sides, the legs have length 1-r. By the Pythagorean Theorem, $(r+1)^2 = (1-r)^2 + (1-r)^2$. Rearrange to $r^2 - 6r + 1 = 0$, and use the quadratic formula to get $r = 3 \pm 2\sqrt{2}$. The positive root is greater than 1, so r = **<u>3 – 2√2</u>**.

6) [2005 Indiana State Math Contest, Geometry/Integrated Math II, #12]



Use the line segment connecting the center of the semicircle and the right hand circle as the hypotenuse of a right triangle. One leg is the radius of the middle circle. The other leg connects the centers of the middle and right hand circles. If we denote the unknown radius to be r, then $r^2 + (2r)^2 = (4-r)^2$. Expand and rearrange to $4(r^2 + 2r - 4) = 0$. The quadratic formula gives $r = -1 \pm \sqrt{5}$. The negative root is negative, so the answer is **<u>-1 + √5</u>**.

7) [2006 TAMU DE #21]

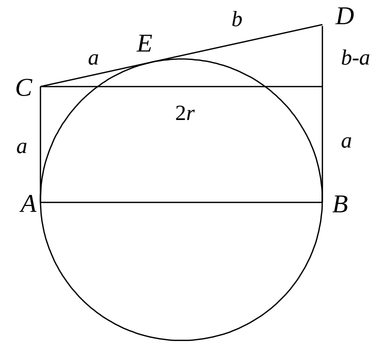


This problem can be solved by constructing two right triangles. The hypotenuse of the first triangle connects the centers of the semicircle and the smaller circle (lower right in diagram). If we denote the radius of the smaller circle to be r, then the hypotenuse has length 4-r. One of the legs of this triangle is a radius of the smaller circle, denoted r, and the other leg is denoted x. Thus, $r^2 + x^2 = (4-r)^2$.

The hypotenuse second right triangle connects the centers of the two circles, for a length of 2+r. The legs are 2-r and x, giving $(2-r)^2 + x^2 = (2+r)^2$.

Solve for x^2 in both equations to eliminate it: $x^2 = (4-r)^2 - r^2 = (2+r)^2 - (2-r)^2$. Expand and solve to give r = **<u>1</u>**.

8) [Modified from NSML Math Contest sample, #3]



Draw the perpendicular from C to BD, forming a rectangle and right triangle. Denote the radius of the circle to be r, CE = a and DE = b. Since tangents to a common point are equal, AC = CE = a and BD = DE = b. Then, the right triangle has legs 2r and b-a and hypotenuse a+b, giving $(2r)^2 + (b-a)^2 = (a+b)^2$. Expand and note that the a^2 and b^2 terms cancel out, and the rest simplifies to $ab = r^2 = 3$. So, the area is πr^2 = **<u>3π</u>**.