Baltimore County ARML Team
Formula Sheet, v2.1 (08 Apr 2008)
By Raymond Cheong

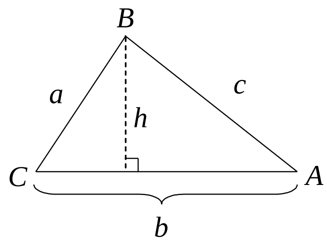
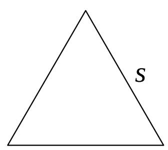
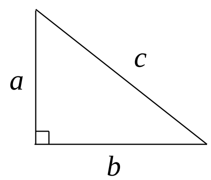
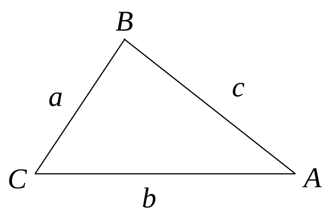
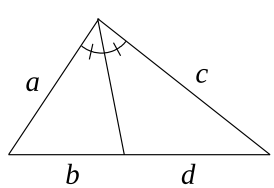
POLYNOMIALS	
Factoring	Difference of squares $a^2 - b^2 = (a+b)(a-b)$ Difference of cubes $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ Sum of cubes $a^3 + b^3 = (a+b)(a^2-ab+b^2)$ Any integer n $a^n - 1 = (a-1)(a^{n-1} + a^{n-2} + ... + a + 1)$ Odd integers n $a^n + 1 = (a+1)(a^{n-1} - a^{n-2} + a^{n-1} - ... - a + 1)$
Binomial expansion	$(a+b)^n = {}_nC_0 a^n + {}_nC_1 a^{n-1}b + {}_nC_2 a^{n-2}b^2 + ... + {}_nC_{n-1} ab^{n-1} + {}_nC_n b^n$
Relationship between roots and coefficients	Quadratics $ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ $(x-r_1)(x-r_2) = x^2 \underbrace{-(r_1+r_2)}_{\text{negative sum of roots}} x + \underbrace{r_1 r_2}_{\text{product of roots}}$ Cubics $(x-r_1)(x-r_2)(x-r_3) =$ $x^3 \underbrace{-(r_1+r_2+r_3)}_{\text{negative sum of roots}} x^2 \underbrace{+(r_1r_2+r_1r_3+r_2r_3)}_{\text{sum of roots taken 2 at a time}} x \underbrace{-r_1r_2r_3}_{\text{negative product of roots}}$ General (Vieta's Theorem) $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1x + a_0 = 0$ $a_{n-1} = -(r_1 + r_2 + ... + r_n)$ $a_{n-2} = r_1r_2 + r_1r_3 + ... + r_{n-1}r_n$ … $a_{n-p} = (-1)^p$ (sum of roots taken p at a time) … $a_0 = (-1)^n r_1 r_2 ... r_n$
Rational root theorem	Let $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1x + a_0 = 0$, where all coefficients are integers. All rational roots (if they exist) are of the form $\pm b/c$ where b and c are factors of a_0 and a_n, respectively.

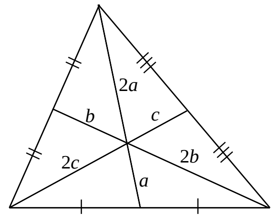
SEQUENCES	
Arithmetic sequences	Consecutive terms have the same difference: $a, a+d, a+2d, a+3d, ..., a+(n-1)d$ (n terms) sum = (# of terms)(average of first and last term) sum = (# of terms)(average of all terms) sum = (# of terms)(median of all terms) $1+2+3+...+n = \frac{n(n+1)}{2}$ first n integers $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ first n odd integers
Geometric sequences	Consecutive terms have the same ratio: $a, ar, ar^2, ar^3, ..., ar^{n-1}$ (n terms) finite sum = $a + ar + ar^2 + ... + ar^{n-1} = \frac{a(1-r^n)}{1-r}$ infinite sum = $a + ar + ar^2 + ... = \frac{a}{1-r}$, $\lvert r \rvert < 1$ sum of powers of 2 = $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$
Other sequences	Sum of squares = $1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$ Sum of cubes = $1^3 + 2^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$

LOGARITHMS	
Basic properties	Definition: $\log_b a = c$ means that $b^c = a$ $b^{\log_b a} = a$ $\log_b 1 = 0$ $\log_b a^n = n \log_b a$ $\log_b b = 1$ $\log_b mn = \log_b m + \log_b n$ $\log_b a = \frac{1}{\log_a b}$ $\log_b \frac{m}{n} = \log_b m - \log_b n$ $\log_b a = \frac{\log_c a}{\log_c b}$

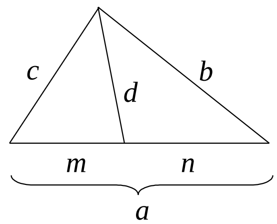
NUMBER THEORY	
Modular arithmetic	$a \equiv b \pmod{m}$ means that a and b leave the same remainder when divided by m If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then for any integer n: $a \pm n \equiv b \pm n \pmod{m}$ $an \equiv bn \pmod{m}$ $a \pm c \equiv b \pm d \pmod{m}$ $ac \equiv bd \pmod{m}$ Fermat's Little Theorem: If p is prime and a is relatively prime to p then $a^{p-1} \equiv 1 \pmod{p}$
Number of factors	If the prime factorization of $n = p_1^{a_1} p_2^{a_2} \dots p_m^{a_m}$, then n has $(a_1+1)(a_2+1)\dots(a_m+1)$ positive factors.
Definition of base	A number with digits a_n, a_{n-1}, …, a_1, a_0 in base b means that $\underline{a_n a_{n-1} \dots a_1 a_0}_b = a_n b^n + a_{n-1} b^{n-1} + \dots + a_1 b + a_0$
Divisibility rules	Let $k = \underline{a_n a_{n-1} \dots a_1 a_0}$ in base 10. 2 last digit (a_0) is even 3 sum of digits ($a_0 + a_1 + \dots + a_n$) is divisible by 3 5 last digit (a_0) is 0 or 5 7 $\underline{a_n a_{n-1} \dots a_1} - 2a_0$ is divisible by 7 (use iteratively) 9 sum of digits ($a_0 + a_1 + \dots + a_n$) is divisible by 9 10 last digit (a_0) is 0 11 $a_0 - a_1 + a_2 - a_3 + a_4 - \dots$ is divisible by 11 2^k number formed by last k digits are divisible by 2^k 10^k last k digits are 0
Remainder rules	Let $k = \underline{a_n a_{n-1} \dots a_1 a_0}$ in base 10. 2, 5, 10 last digit has same remainder 3, 9 sum of digits has same remainder 11 $a_0 - a_1 + a_2 - a_3 + a_4 - \dots$ has same remainder 2^k, 10^k number formed by last k digits has same remainder

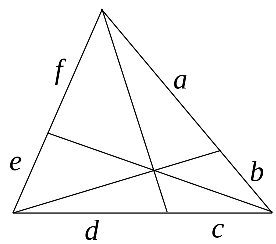
Combinatorics	<u>Permutation</u>: number of ways to choose *r* items from *n* distinct objects where different orderings are distinct $\quad {}_nP_r = \frac{n!}{r!}$ <u>Combination</u>: number of ways to choose *r* items from *n* distinct objects where order does not matter $\quad {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$ <u>Partition</u>: number of ways to group *n* identical objects into *m* distinct bins, with zero items in a bin allowed $\quad \binom{n+m-1}{m-1}$ <u>Word rearrangement</u>: Number of ways to rearrange the letters of a word with n_A A's, n_B B's, …, n_Z Z's, and *n* letters in total. $\quad \frac{n!}{n_A!n_B!n_C!...n_Z!}$
Pascal's triangle	Ones down the right and left sides. Each entry is the sum of the two entries above it. Sum of the n^{th} row = 2^n Each entry is a combination. Entries of the n^{th} row give the coefficients of the n^{th} order binomial expansion. (The rows are numbered off starting from 0.) 1 1 1 1 2 1 1 3 3 1 1 4 6 4 1 1 5 10 10 5 1 1 6 15 20 15 6 1
Prime factorization of years	$1936 = 44^2 = 2^4 \cdot 11^2$ $2007 = 3^2 \cdot 223$ $2008 = 2^3 \cdot 251$ $2009 = 7^2 \cdot 41$ $2010 = 2 \cdot 3 \cdot 5 \cdot 67$ $2025 = 45^2 = 3^4 \cdot 5^2$

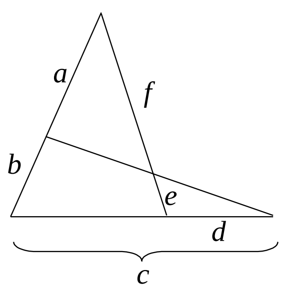
TRIANGLE GEOMETRY	
Area	 area = $\frac{1}{2}bh$ area = $\frac{1}{2}ab\sin C$ area = $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$ (semiperimeter)  Equilateral triangle Area = $s^2 \frac{\sqrt{3}}{4}$
Pythagorean Theorem	 $a^2 + b^2 = c^2$ Common triples: 3-4-5, 5-12-13, 7-20-21, 9-40-41, and multiples (e.g. 6-8-10) m^2-n^2, 2mn, m^2+n^2 where *m*,*n* are integers is a Pythagorean Triple
Trigonometric laws of triangles	 Law of sines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ Law of cosines: $c^2 = a^2 + b^2 - 2ab\cos C$ Law of tangents: $\tan A \tan B \tan C = \tan A + \tan B + \tan C$
Angle bisectors	 $\frac{a}{b} = \frac{c}{d}$

<table>
<tr>
<td>Medians</td>
<td>


</td>
<td>Medians divide each other into 2:1 segments

The 6 little triangles all have equal area.</td>
</tr>
<tr>
<td>Famous triangle theorems</td>
<td>


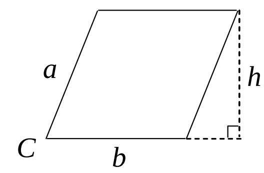
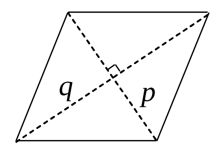
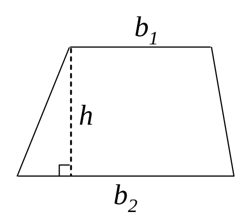
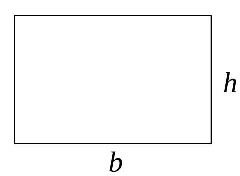
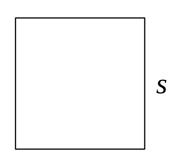
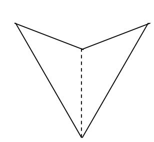
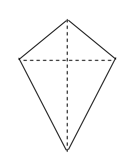


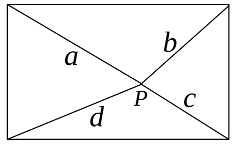
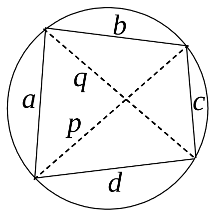


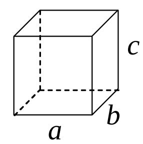
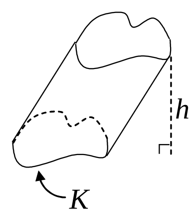
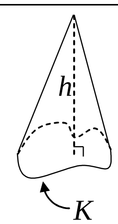
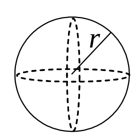
</td>
<td>Stewart’s Theorem
$amn + ad^2 = mb^2 + nc^2$

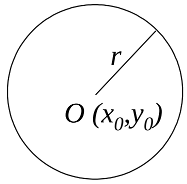
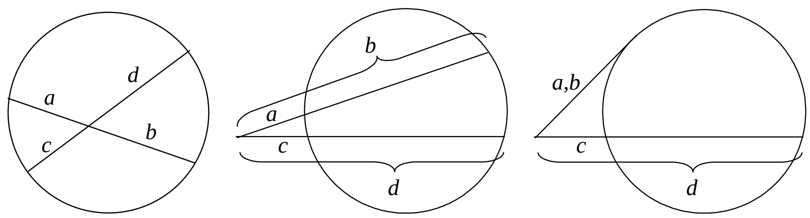
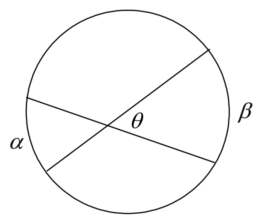
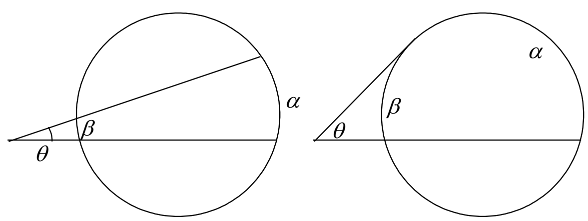
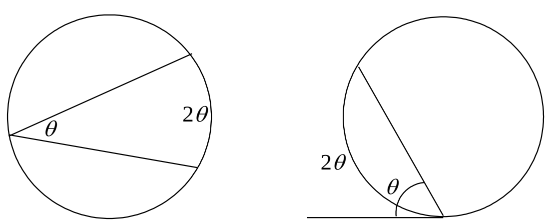
Ceva’s Theorem
$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right)\left(\frac{e}{f}\right) = 1$

Menelaus’ Theorem
$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right)\left(\frac{e}{f}\right) = 1$</td>
</tr>
</table>

QUADRILATERAL GEOMETRY	
Parallelogram	<u>Def.</u>: opposite sides are parallel. <u>Properties</u>: - Opposite sides have equal lengths - Opposite angles are congruent - Diagonals bisect each other - Diagonals form two pairs of similar triangles Area = $bh = ab\sin C$ 
Rhombus	<u>Def.</u>: parallelogram w/ all equal sides <u>Properties</u>: - Diagonals p and q are perpendicular. - Diagonals form 4 congruent triangles Area = ½ pq 
Trapezoid	<u>Def.</u>: 1 pair of opposite sides are parallel <u>Properties</u>: The two triangles formed by the diagonals and bases are similar. Area = $\frac{1}{2}(b_1 + b_2)h$ 
Rectangle	<u>Def.</u>: parallelogram with all right angles <u>Properties</u>: Highly symmetric Area = bh 
Square	<u>Def.</u>: rectangle with all sides equal <u>Properties</u>: Highly symmetric Area = s^2 
Other	Chevron (left): Symmetric and concave Kite (right): Perpendicular diagonals, one of which bisects the other  

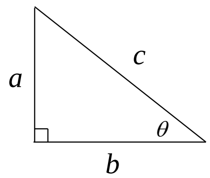
British Flag Theorem		Any point P (inside, outside, on, above, or below rectangle): $a^2 + c^2 = b^2 + d^2$
Cyclic quadrilaterals		$ac + bd = pq$ (Ptolemy's Theorem) area = $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ where $s = \dfrac{a+b+c+d}{2}$ (semiperimeter)

3D & POLYGON GEOMETRY		
Rectangular parallelpiped	Volume = abc Internal diagonal = $\sqrt{a^2 + b^2 + c^2}$	
Generalized cylinder	Volume = Kh (K is the area of the base)	
Generalized pyramid	Volume = $\dfrac{1}{3}Kh$ (K is the area of the base)	
Sphere	Volume = $\dfrac{4}{3}\pi r^3$ Surface area = $4\pi r^2$	
Polygons	Sum of interior angles = $180°(n-2)$ (n-sided convex polygon) Area = $\dfrac{s^2 n}{4\tan(180^\circ / n)}$ (n-sided regular polygon)	

CIRCLE GEOMETRY	
Basic properties of circles	 Area = πr^2 Circumference = $2\pi r$ Equation: $(x-x_0)^2+(y-y_0)^2=r^2$
Power of a point theorem	 In each diagram, *ab* = *cd*
Angles formed by chords, secants, and tangents	 $\theta=\dfrac{\alpha+\beta}{2}$  $\theta=\dfrac{\alpha-\beta}{2}$  Angle is half the arc formed

TRIGONOMETRY

Definitions



π radians = 180º

$$\sin\theta = \frac{a}{c} \qquad \csc\theta = \frac{1}{\sin\theta}$$

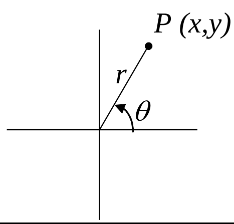
$$\cos\theta = \frac{b}{c} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{a}{b} = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Common angles

	0º	30º	45º	60º	90º	180º
sin θ	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	0
cos θ	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	-1
tan θ	0	$\sqrt{3}/3$	1	$\sqrt{3}$	∞ or -∞	0

Polar vs. Cartesian coordinates



$$x = r\cos\theta \qquad \tan\theta = y/x$$

$$y = r\sin\theta \qquad r = \sqrt{x^2 + y^2}$$

Pythagorean Theorem

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \cot^2\theta = \csc^2\theta \qquad \tan^2\theta + 1 = \sec^2\theta$$

Double angle formulas

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Sum and difference formulas

$$\sin(a+b) = \sin a\cos b + \cos a\sin b$$

$$\sin(a-b) = \sin a\cos b - \cos a\sin b$$

$$\cos(a+b) = \cos a\cos b - \sin a\sin b$$

$$\cos(a-b) = \cos a\cos b + \sin a\sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a\tan b} \qquad \tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a\tan b}$$